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Localization of relativistic systems

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Abstract. We investigate Poincaré-covariant phase spaces which admit space-time localization and cases when this localization can be chosen commutative.

0. Introduction

The notion of a localization (a covariant position) introduces the space-time context into abstract Poincaré-covariant phase spaces.

Coadjoint orbits of the Poincaré group—'minimal' Poincaré-covariant phase spaces—were recognized as spaces of motions of classical elementary relativistic particles some time ago [1]. They do not admit any localization (cf section 1). Recently, extended phase spaces for such particles were introduced [2] as 'minimal' Poincaré-covariant phase spaces with commutative (see below) localization.

In [3, 4] (see also [5]) it was pointed out that in the product of two-twistor phase spaces, there is naturally defined localization (one has to exclude parallel momenta). It is non-commutative: the Poisson bracket of two space-time coordinates is proportional to the (internal) spin tensor. In [5] it was shown that in the two-twistor phase space commutative localization can also be found (hence this space 'decomposes' on some extended phase spaces).

The twistor phase space provides an elegant description of (all 'physical' [1]) massless Poincaré-coadjoint orbits. The question now arises about the existence of a (commutative) localization for the product of two arbitrary ('physical') coadjoint orbits. By a simple geometric construction, one shows that such a localization indeed exists (after exclusion of parallel momenta). In this paper we describe this construction in detail for the case of spinless orbits and we show that one can find a commutative localization if one of the particles is massless.

Throughout this paper, $\mathcal{M}$ denotes the Minkowski space-time, G denotes the (connected component of unity of the) Poincaré group and $\mathfrak{g}$—its Lie algebra.

We denote by V the subgroup of translations in the Poincaré group G. This is a normal subgroup and

$$L := G/V$$

is the Lorentz group, acting naturally in V—the tangent space of $\mathcal{M}$. Any choice of $x \in \mathcal{M}$ allows us to identify L with the stabilizing subgroup G_x of G. We denote by $\mathfrak{l}$ and $\mathfrak{g}_x$ the Lie algebras corresponding to L and G_x.

A choice of a basis e_k in V allows us to identify the generators of $\mathfrak{l}$ as

$$M_{kl} := e_k \otimes g(e_l) - e_l \otimes g(e_k) \in \mathfrak{l} \subset \mathrm{End}\, V$$

where g denotes the metric tensor (here considered as a map $g(\cdot) : V \to V^*$). Choosing $x_0 \in \mathcal{M}$ we can identify $\mathfrak{g}$ as the semidirect product $\mathfrak{g} \simeq V \rtimes \mathfrak{l}$, with (the 'right'—see [2]) commutators given by

$$[M_{jk}, M_{ln}] = g_{jl} M_{kn} + g_{kn} M_{jl} - g_{jn} M_{kl} - g_{kl} M_{jn} \qquad [M_{jk}, e_l] = g_{jl} e_k - g_{kl} e_j$$

and $[e_j, e_k] = 0$. We recall that the same formulae define the Poisson brackets on $\mathfrak{g}^*$:

$$\{M_{jk}, M_{ln}\} = g_{jl} M_{kn} + g_{kn} M_{jl} - g_{jn} M_{kl} - g_{kl} M_{jn} \qquad \{M_{jk}, p_l\} = g_{jl} p_k - g_{kl} p_j \tag{1}$$

and $\{p_j, p_k\} = 0$ (we have changed the notation for e_k using the traditional notation p_k for the linear momentum).

1. Relativistic systems and localization

Isolated relativistic system (IRS) is a symplectic manifold P together with a symplectic action of G on it. (We identify an IRS with the underlying symplectic manifold P, if the action of G is assumed to be known.) Due to the known properties of G ($[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, $H^1(\mathfrak{g}, \mathfrak{g}^*) = \{0\}$, $(\mathfrak{g}^*)_{\mathrm{inv}} = \{0\}$), there is exactly one equivariant momentum mapping $J : P \to \mathfrak{g}^*$ for this action, hence each IRS is a Hamiltonian G-space (in a unique way). (Recall that J is equivariant if and only if its components (p_j, M_{kl}) satisfy (1).)

Examples of IRS.

(1) Coadjoint orbits of G (Souriau's spaces of motions of elementary systems).
(2) Extended phase spaces P_z of elementary systems (see [2] and below).
(3) Twistor phase space T (see [6, 4, 5]).
(4) Products of IRSs.

Let P be an IRS. A *localization of* P is a G-equivariant map $x : P \to \mathcal{M}$. An IRS is *localizable* (it is a LIRS) if it admits a localization.

Using a basis e_k in V and choosing an origin $x_0 \in \mathcal{M}$, a localization x is equivalently described by four functions x^j ($j = 0, \ldots, 3$) such that

$$\{p_j, x^k\} = \delta_j{}^k \qquad \{M_{jk}, x^l\} = \delta_j{}^l x_k - \delta_k{}^l x_j. \tag{2}$$

Examples of IRS not admitting any localization.

(1) Coadjoint orbit of G.

Proof. The existence of x^k satisfying (2) yields $p^k = \frac{1}{2}\{p^2, x^k\} = 0$ (since $p^2 := g^{jk} p_j p_k$ is constant on the orbit) and implies the contradiction $\delta_k{}^j = \{p_k, x^j\} = 0$. □

(2) Any IRS on which $m^2 := p^2$ is fixed (proof as above).
(3) Twistor space T (the mass is fixed $m = 0$).

Examples of IRS admitting a localization.

(1) Extended phase spaces (EPS) [2]; we recall the definition at the end of this section.
(2) Fixed spin reduction of EPS, cf [2].
(3) Product of a LIRS by an IRS.

(4) Product of two 'physical' coadjoint orbits (excluding parallel momenta), cf section 2 below.

(5) Two-twistor space (excluding parallel momenta) [7, 4, 5].

A localization is *commutative* if $\{x^j, x^k\} = 0$, i.e. if $x : P \to \mathcal{M}$ is a Poisson map (where $\mathcal{M}$ has the trivial Poisson structure).

A commutative localization is *complete* if x is a complete Poisson map, i.e. Hamiltonian vector fields of functions x^j are complete.

In [2] it was proven that in the case of a commutative localization x, the (Lorentz) momentum with respect to x,

$$\Omega_{jk} = M_{jk} - p_j x_k + p_k x_j$$

commutes with x and p. More generally, assuming nothing about the commutation properties of x, we have

$$\{\Omega_{jk}, p_l\} = 0 \qquad \{\Omega_{jk}, x^l\} = \{x_j, x^l\}p_k - \{x_k, x^l\}p_j. \tag{3}$$

One can easily show that $\{x^j, x^k\} = 0$ is equivalent to $\{\Omega_{jk}, x^l\} = 0$.

An *extended phase space* (EPS) is an IRS with complete commutative localization, such that the group G together with the Hamiltonian vector fields of x^j acts transitively on P. Such spaces were introduced in [2] in order to formulate the (free) dynamics of elementary systems. It was shown in [2] (using (3)) that extended phase spaces are all of the form

$$P = T^*\mathcal{M} \times \mathcal{O}$$

where $\mathcal{O}$ is a coadjoint orbit in $\mathfrak{l}^*$. Hence $P = T^*\mathcal{M}$ or

$$P = P_z = T^*\mathcal{M} \times \mathcal{O}_z$$

where $z = a + \mathrm{i}b$ is the parameter of the orbit:

$$\mathcal{O}_z := \{\Omega \in \mathfrak{l} \setminus \{0\} : \langle \Omega, \Omega \rangle_{\mathbb{C}} = z^2\}$$

(we have identified $\mathfrak{g}^*$ with $\mathfrak{g}$ using the Killing form $\langle \Omega, \Omega \rangle := -\frac{1}{2}\,\mathrm{tr}\,\Omega^2$ and denoted by $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ the corresponding complex Killing form on $\mathfrak{l} \equiv sl(2, \mathbb{C})$).

2. Products of two spinless orbits

Such orbits are symplectic reductions of $T^*\mathcal{M}$ with respect to a fixed mass. Instead of showing that any product of two such orbits (excluding parallel momenta) has a localization, we shall show that there exists a localization of $P := T^*\mathcal{M} \times T^*\mathcal{M}$ (parallel momenta excluded) which depends only on the reduced variables. The construction of a map $x : P \to \mathcal{M}$ is fairly simple. Given (x_1, p_1, x_2, p_2), where $x_1, x_2 \in \mathcal{M}$, $p_1, p_2 \in V^* \cong V$, consider the corresponding two world lines,

$$X_1(t) = x_1 + tp_1 \qquad X_2(s) = x_2 + sp_2$$

where we relate the parameter s to t in such a way that $X_1(t) - X_2(s)$ is perpendicular to the total momentum $p := p_1 + p_2$. This means that $s = c + dt$ where

$$c = \frac{(x_1 - x_2) \cdot p}{p_2 \cdot p} \qquad d = \frac{p_1 \cdot p}{p_2 \cdot p}$$

(the dot denotes the scalar product). It is easy to see that there exists exactly one t such that the distance from $X_1(t)$ to $X_2(s(t))$ is minimal, namely:

$$t = t_0 = -\frac{y \cdot q}{q \cdot q} \qquad y = [(x_1 - x_2) \wedge_g p_2]p \qquad q := (p_1 \wedge_g p_2)p \tag{4}$$

where by $u \wedge_g v$ we denote (cf [2]) the element of $\mathfrak{l} \subset \mathrm{End}\, V$ which maps $w \in V$ to

$$(u \wedge_g v)w := u(v \cdot w) - v(u \cdot w).$$

In other words, two (not parallel) free motions determine uniquely a pair of points on the corresponding world lines, which realizes the shortest distance (in the co-moving frame) between the two lines. Any fixed affine combination of these two points,

$$(1 - \lambda)X_1(t_0) + \lambda X_2(s(t_0)) \tag{5}$$

(λ is a fixed number), gives a localization. A simple example is given by the first point ($\lambda = 0$):

$$x := X_1(t_0) = x_1 + t_0 p_1 \tag{6}$$

where t_0 is given by (4).

Proposition 2.1. The localization given by (6) is commutative if and only if $m_2^2 := p_2^2 = 0$ (the second particle is massless).

Proof. We have

$$\{x_1 + t_0 p_1 \overset{\otimes}{,} x_1 + t_0 p_1\} = \{x_1, t_0\} \wedge p_1 + t_0\{p_1, t_0\} \wedge p_1 \tag{7}$$

(tensor notation). Since $\{p_1, y \cdot q\} = (q \wedge p)p_2$, it is easy to see that

$$\{p_1, t_0\} \wedge p_1 = -\frac{p^2}{q^2}(p_1 \cdot p_2)p_1 \wedge p_2. \tag{8}$$

Now, for the commutativity $\{x \overset{\otimes}{,} x\} = 0$, the component of $\{x_1, t_0\}$ which is proportional to $\Delta x := x_1 - x_2$ must vanish. This component comes only from $\{x_1, y \cdot q\}$. The coefficient at Δx in the latter expression is given by

$$p_2 \cdot q - (p_2 \cdot p)^2 = -p^2 p_2^2$$

hence $p_2^2 = 0$.

Assuming now that $p_2^2 = 0$, we have

$$t_0 = -\frac{\Delta x \cdot p_2}{p_1 \cdot p_2} \tag{9}$$

and one can easily calculate the brackets

$$\{p_1, t_0\} = -\frac{1}{p_1 \cdot p_2} p_2 \qquad \{x_1, t_0\} = -\frac{\Delta x \cdot p_2}{p_1 \cdot p_2} p_2 = t_0 p_2. \tag{10}$$

Substituting this into (7) gives immediately the result. □

Note that the Lorentz momentum with respect to x is given by

$$\Omega = p_1 \wedge x_1 + p_2 \wedge x_2 - (p_1 + p_2) \wedge x = p_2 \wedge (x_2 - x). \tag{11}$$

For $p_2^2 = 0$, it follows that Ω is simple and $\frac{1}{2}\,\mathrm{tr}\,\Omega^2 = 0$, hence we obtain the nilpotent Lorentz orbit ($a = b = 0$ in [2]). This shows that the product of any (spinless) orbit by the massless one is in fact (contained in) the extended phase space $P_0 = T^*\mathcal{M} \times T^*S^2$. The 'celestial sphere' is here provided by the projective part of p_2. A direct calculation shows that

$$\{p_2 \overset{\otimes}{,} x\} = \frac{1}{p_1 \cdot p_2} p_2 \otimes p_1 \tag{12}$$

which ensures the commutativity of x with the projective part of p_2.

Remark. It would be interesting to find a commutative localization for the case when both particles are massive. Let us also note that among all possible localizations (5) there is a distinguished one—that lies on the world line of the total system:

$$X := X(t_0) = \frac{(p_1 p)X_1(t_0) + (p_2 p)X_2(s(t_0))}{p^2} \tag{13}$$

(one can easily show that the world line of the total system is the following 'affine combination' of the world lines of the components: $X(t) = \frac{(p_1 p)X_1(t)+(p_2 p)X_2(s(t))}{p^2}$). It is characterized by the condition that the Lorentz momentum with respect to X applied to p gives zero.

Using (11) we can calculate the spin function on P. Since

$$\tfrac{1}{2}\,\mathrm{tr}\,\Omega^2 = -p_2^2 \cdot (x_2 - x)^2 \qquad \Omega p = p_2[(x_2 - x)\cdot p)] - (x_2 - x)(p_2 \cdot p) \tag{14}$$

and

$$(\Omega p)^2 = p_2^2[(x_2 - x)\cdot p)]^2 + (x_2 - x)^2(p_2 \cdot p)^2 \tag{15}$$

we have

$$s^2 = -\frac{(\Omega p)^2}{p^2} - \frac{1}{2}\,\mathrm{tr}\,\Omega^2 = (x_2 - x)^2\frac{[p_2^2 p^2 - (p_2\cdot p)^2]}{p^2} - p_2^2\frac{[(x_2 - x)\cdot p]^2}{p^2}. \tag{16}$$

In particular, if $p_2^2 = 0$, we have

$$s^2 = -(x_2 - x)^2\frac{(p_2\cdot p)^2}{p^2} = -p^2\left[\frac{(p_2\cdot p)(X_2(s(t_0)) - x)}{p^2}\right]^2 \tag{17}$$

and since $x - X = \frac{(p_2\cdot p)(x - X_2(s(t_0)))}{p^2}$,

$$s^2 = m^2 r^2 \tag{18}$$

where $r^2 = -(x - X)^2$.

3. Two-twistor phase space

Two twistors with non-parallel momenta determine a two-dimensional subspace in the twistor space (transversal to the distinguished 'spinor' subspace), hence a point in the complexified Minkowski space. Due to the possibility of taking the 'real part' of points in the complexified Minkowski space (we explain this construction in the appendix), one obtains a localization in the two-twistor phase space [3–5]. It is not commutative, namely the Poisson brackets of space-time coordinates is given by

$$\{X^j, X^k\} = -\frac{1}{m^2}R^{jk} \tag{19}$$

where $m^2 = p^2$, p is the total linear momentum and R^{jk} is the rotational (with respect to p) part of the total Lorentz angular momentum. This remarkable formula coincides with that one for the spin-invariant localization, considered in [2] (the localization which descends to the fixed spin reduced space). A reasonable conjecture about the localization given by (13) is that it has the same property.

In [5] it was shown that in the two-twistor phase space there exists also a commutative localization. It is obtained by considering the 'world line' of, say, the first twistor, with respect to the common rest frame (so the result is similar to proposition 2.1). The commutativity was proved using the basic twistor calculus. However, the (commutative as well as non-commutative) localization is in fact a function of the coadjoint orbit variables (p_1, M_1), (p_2, M_2) as can be seen from formula (32) of [5]. This means that

(1) the localization facts concerning the two-twistor space can be proved using only products of coadjoint orbits (of massless particles with spin);

(2) these facts are indeed valid for such orbits.

Appendix. The complexified Minkowski space

Let T be the twistor space (four-dimensional complex vector space equipped with a Hermitian form of signature $++--$) and let S denote the distinguished isotropic two-dimensional ('spinor') subspace. The Minkowski space $\mathcal{M}$ is defined as the set of all complex two-dimensional isotropic subspaces of T which are transversal to S. The complexified Minkowski space $\mathcal{M}^{\mathbb{C}}$ is identified as the set of all complex two-dimensional subspaces of T which are transversal to S. There is a one-to-one correspondence between $z \in \mathcal{M}^{\mathbb{C}}$ and projection operators $P^2 = P$ acting in T whose image coincides with S, such that

$$z = \ker P.$$

It is easy to see that the natural (antiholomorphic) involution in $\mathcal{M}^{\mathbb{C}}$ given by $z \mapsto z^{\perp}$ is implemented by the operation

$$P \mapsto I - P^*$$

on projections (here * denotes the Hermitian conjugation with respect to the given form). Indeed, $I - P^*$ is a projection with the required properties:

$$\begin{aligned} \mathrm{im}(I - P^*) &= \ker P^* = (\mathrm{im}\, P)^{\perp} = S^{\perp} = S \\ \ker(I - P^*) &= \mathrm{im}\, P^* = (\ker P)^{\perp}. \end{aligned}$$

Of course, points $z \in \mathcal{M}^{\mathbb{C}}$ which belong to $\mathcal{M}$ (are real) correspond to projections P satisfying

$$P = I - P^*. \tag{20}$$

Now, for any projection P (corresponding to some $z \in \mathcal{M}^{\mathbb{C}}$), we can define the *real part*,

$$\mathrm{Re}\, P := \tfrac{1}{2}(P + I - P^*).$$

It is a projection: $(\mathrm{Re}\, P)^2 = \mathrm{Re}\, P$. Indeed, since $P^*P = 0$, we have

$$\begin{aligned} (\mathrm{Re}\, P)^2 &= \tfrac{1}{4}[P^2 + (I - P^*)^2 + P(I - P^*) + (I - P^*)P] \\ &= \tfrac{1}{4}[P + (I - P^*) + (I - P^*) + P]. \end{aligned}$$

The image of $\mathrm{Re}\, P$ is S. Moreover, $\mathrm{Re}\, P$ satisfies (20). It follows that $\mathrm{Re}\, P$ corresponds to a point in $\mathcal{M}$ which we denote by $\mathrm{Re}\, z$ (the *real part* of $z \in \mathcal{M}^{\mathbb{C}}$). Geometrically, $\mathrm{Re}\, z$ is the 2-dimensional subspace which is 'half-way' from z to $z^{\perp}$ (in the direction of S).

The *imaginary part*,

$$\mathrm{Im}\, P = \frac{1}{2\mathrm{i}}(P - I + P^*)$$

has the properties

$$(\mathrm{Im}\, P)|_S = 0 \qquad \mathrm{im}(\mathrm{Im}\, P) \subset S \qquad (\mathrm{Im}\, P)^* = -\,\mathrm{Im}\, P$$

hence $T \ni Z \mapsto Z + (\mathrm{Im}\, P)Z = Z + \mathrm{i}v(\pi(Z))$ is the action of a translation by a (real) vector $v \in S \otimes \overline{S}$ (cf [5]). Here $\pi(Z)$ is the projection of Z on T/S and v is the linear map from T/S to S, defined by $\frac{1}{i}(\mathrm{Im}\,P)$. We set $\mathrm{Im}\,z := v$, if P is the projection corresponding to $z \in \mathcal{M}^{\mathbb{C}}$. Geometrically, $\mathrm{Im}\,z$ is the 'difference' (in the direction of S) of z and $z^{\perp}$ (scaled by the imaginary unit).

The above structure of $\mathcal{M}^{\mathbb{C}}$ agrees with the obvious intuition that the complexification of an affine space is modelled by its tangent bundle.

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